

# Maximally Supersymmetric $\text{AdS}_4$ Vacua in $N = 4$ Supergravity

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## ABSTRACT

We study AdS backgrounds of  $N = 4$  supergravity in four space-time dimensions which preserve all sixteen supercharges. We show that the graviphotons have to form a subgroup of the gauge group that consists of an electric and a magnetic  $\text{SO}(3)_+ \times \text{SO}(3)_-$ . Moreover, these  $N = 4$  AdS backgrounds are necessarily isolated points in field space which have no moduli.

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## 1 Introduction

The maximally symmetric space-time backgrounds of supergravity theories which preserve some of the supercharges are either anti-de Sitter (AdS) or Minkowski (M) spaces. It is of interest to study such backgrounds and find the (model-independent) properties of the associated moduli spaces. In  $N = 2$  supergravities in four space-time dimensions ( $d = 4$ ) the fully supersymmetric  $\text{AdS}_4$  backgrounds were determined in [1, 2] while the structure of the moduli space of  $N = 1$  and  $N = 2$   $\text{AdS}_4$  backgrounds was given in [3]. It was found that generically a supersymmetric  $\text{AdS}_4$  background of  $N = 1$  and  $N = 2$  supergravity has no moduli space. However, by appropriately tuning the mass parameters of the theory flat directions which preserve all supercharges may occur. In  $N = 1$  they span a field space which is necessarily real and has at best half the dimension of the original field space. In  $N = 2$  the moduli space is a Kähler manifold – again at best of half the dimension of the original field space. Both results are in agreement with the AdS/CFT correspondence which relates these backgrounds to superconformal field theories on the  $d = 3$  boundary of  $\text{AdS}_4$  with multiplets which are in representations of theories that have only half of the supercharges.<sup>1</sup>

For  $N = 4$  supergravity in  $d = 4$  an analogous investigation is lacking so far and it is the purpose of this paper to close this gap. In contrast to gauged supergravities with eight or less supercharges, in  $N = 4$  supergravity the mass parameters cannot be freely tuned and are determined by the choice of the gauge group. This in turn suggests that the dimension

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<sup>1</sup>The other half are the superconformal supercharges.

and structure of the moduli space is also fixed. From the AdS/CFT perspective one expects the moduli space to be hyper-Kähler – if it exists at all.<sup>2</sup>

After the initial construction of (electrically) gauged  $N = 4$  supergravity [5–9] it was shown that within this class of theories no supersymmetric AdS-backgrounds exist and any background preserving some supercharges has to be Minkowskian  $M_4$  [10, 11].<sup>3</sup> However the same papers realized that supersymmetric AdS-backgrounds can occur when additional parameters are non-trivial. These de Roo-Wagemans angles gauge isometries with respect to dual magnetic vector multiplets. Generic gauged supergravities including magnetic vector multiplets have been constructed in [13] introducing what is now called the embedding tensor formalism. This was subsequently used in [14] to construct the most general gauged  $N = 4$  supergravity coupled to vector multiplets in  $d = 4$ . It is within this framework that we conduct our analysis.

We find that the existence of a fully supersymmetric  $AdS_4$  background imposes a set of constraints on the embedding tensor. They in turn imply that the complex scalar  $\tau$  of the gravitational multiplet has to be uncharged and they also restrict the possible gauge groups  $G_0$ . More precisely, the six graviphotons of  $N = 4$  supergravity have to gauge an unbroken  $SO(3)_+ \times SO(3)_-$  inside the R-symmetry  $SO(6)_R$  where one of the factors is electric while the other is magnetic. In general the two factors can be part of a larger gauge group with the structure  $G_0 = G_+ \times G_- \times G_0^v \subset SO(6, n)$  where  $G_0^v \subset SO(n)$  is a separate factor. In the  $N = 4$   $AdS_4$  vacuum the group  $G_+ \times G_-$  is spontaneously broken to its maximal compact subgroup, containing the two  $SO(3)_\pm$  factors. Furthermore, the potential has supersymmetric flat directions which, however, are precisely the Goldstone bosons of the spontaneous symmetry breaking. No further flat directions and thus no moduli space exists.

This paper is organized as follows. In Section 2 we recall the properties of  $N = 4$  gauged supergravity that we need for our analysis. In Section 3 we analyze  $N = 4$   $AdS_4$  backgrounds and determine the constraints on the embedding tensor. We then show that an  $SO(3)_+ \times SO(3)_-$  subgroup of the R-symmetry group is necessarily gauged and we also determine the allowed structure of the full gauge group  $G_0$ . In Section 4 we show that the conditions for an  $N = 4$  AdS-background admit a set of flat directions corresponding to the Goldstone bosons of the spontaneously broken  $G_0$ . However, no further flat directions do exist which indeed confirms that the backgrounds found in Section 3 are isolated points in the scalar field space. Some of the technical analysis is relegated to three appendices.

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<sup>2</sup>It has been conjectured [4] that  $d = 3$  superconformal field theories with eight supercharges (plus eight conformal supercharges) have no exactly marginal deformations which would in turn suggest that there is no moduli space in an  $N = 4$   $AdS_4$  bulk supergravity. We thank O. Aharony for this inspirational remark which prompted the present investigation.

<sup>3</sup>For a recent analysis of partial  $N = 4$  breaking see, for example, [12] and references therein.

## 2 Preliminaries: $N = 4$ gauged supergravity

Let us set the stage and recall the properties of  $d = 4, N = 4$  gauged supergravity [14] which are relevant in the following. A generic  $N = 4$  spectrum consists of the gravity multiplet together with  $n$  vector multiplets. The gravity multiplet contains the graviton  $g_{\mu\nu}$ , four gravitini  $\psi_\mu^i$ ,  $i = 1, \dots, 4$ , six vectors  $A_\mu^m$ ,  $m = 1, \dots, 6$ , four spin-1/2 fermions  $\chi^i$  and the complex scalar  $\tau$ . We label the vector multiplets with the index  $a = 1, \dots, n$  and each multiplet contains a vector  $A_\mu^a$ , four spin-1/2 gauginos  $\lambda^{ai}$  and 6 scalars  $\phi^{am}$ . So altogether the spectrum features the graviton, 4 gravitini,  $(6+n)$  vector bosons,  $(4+4n)$  spin-1/2 fermions and  $(6n+2)$  scalars.

The field space  $\mathcal{M}$  of the scalars is the coset

$$\mathcal{M} = \frac{SL(2)}{SO(2)} \times \frac{SO(6, n)}{SO(6) \times SO(n)} , \quad (2.1)$$

where the first factor is spanned by  $\tau$  while the second factor is spanned by the scalars  $\phi^{am}$  in the vector multiplets. Both cosets are conveniently parametrized by vielbein fields. For the first factor the vielbein is the complex vector  $\nu_\alpha$ ,  $\alpha = +, -$ , which reads in terms of  $\tau$  as

$$\nu_\alpha = \frac{1}{\sqrt{\text{Im } \tau}} \begin{pmatrix} \tau \\ 1 \end{pmatrix} , \quad (2.2)$$

and defines

$$M_{\alpha\beta} = \text{Re}(\nu_\alpha(\nu_\beta)^*) , \quad \epsilon_{\alpha\beta} = \text{Im}(\nu_\alpha(\nu_\beta)^*) . \quad (2.3)$$

The second factor in (2.1) is parametrized by the vielbein  $\nu = (\nu_M^m, \nu_M^a)$ ,  $M = 1, \dots, n+6$  which is an element of  $SO(6, n)$  and thus obeys

$$\eta_{MN} = -\nu_M^m \nu_N^m + \nu_M^a \nu_N^a , \quad (2.4)$$

where  $\eta_{MN} = \text{diag}(-1, -1, -1, -1, -1, -1, +1, \dots, +1)$  is the flat  $SO(6, n)$  metric. The metric on the coset is then given by

$$M_{MN} = \nu_M^m \nu_N^m + \nu_M^a \nu_N^a = 2\nu_M^m \nu_N^m + \eta_{MN} . \quad (2.5)$$

The couplings of  $N = 4$  gauged supergravity depend on two field-independent  $SL(2) \times SO(6, n)$ -tensors (called embedding tensors) denoted by  $\xi_{\alpha M}$  and  $f_{\alpha[MNP]}$ . Their entries are real numbers and supersymmetry imposes a set of coupled consistency conditions on both tensors known as the quadratic constraints [14]

$$\begin{aligned} \xi_\alpha^M \xi_{\beta M} &= 0 , & \epsilon^{\alpha\beta} (\xi_\alpha^P f_{\beta PMN} + \xi_{\alpha M} \xi_{\beta N}) &= 0 , \\ \xi_{(\alpha}^P f_{\beta) PMN} &= 0 , & 3f_{\alpha R[MN} f_{|\beta| PQ]}^R + 2\xi_{[\alpha[M} f_{\beta] NPQ]} &= 0 , \\ \epsilon^{\alpha\beta} (f_{\alpha MNR} f_{\beta PQ}^R - \xi_\alpha^R f_{\beta R[M[P} \eta_{Q]N]} - \xi_{\alpha[M} f_{|\beta| N] PQ} + \xi_{\alpha[P} f_{|\beta| Q] MN}) &= 0 . \end{aligned} \quad (2.6)$$

Their solutions parametrize the different consistent  $N = 4$  theories and in particular determine the gauge group, the order parameters for spontaneous supersymmetry breaking and the potential.

The full bosonic Lagrangian is recorded in [14] but for the analysis in this paper we only need the potential  $V$  and the kinetic terms of the scalar fields which are given by

$$e^{-1}\mathcal{L} = \frac{1}{16}(D_\mu M_{MN})(D^\mu M^{MN}) + \frac{1}{8}(D_\mu M_{\alpha\beta})(D^\mu M^{\alpha\beta}) - V(M, \xi, f) + \dots \quad (2.7)$$

The covariant derivative of  $M_{MN}$  reads

$$D_\mu M_{MN} = \partial_\mu M_{MN} + 2A_\mu^{P\alpha}\Theta_{\alpha P(M}{}^Q M_{N)Q} \ , \quad (2.8)$$

where  $\Theta_{\alpha PM}{}^Q = f_{\alpha MNP} - \xi_{\alpha[N}\eta_{P]M}$  is the matrix of gauge charges and  $A_\mu^{P+}$  are  $n + 6$  electric gauge bosons while  $A_\mu^{P-}$  are their magnetic duals.<sup>4</sup> We see that a non-vanishing  $\Theta_-$  leads to magnetically charged scalar fields but the above mentioned quadratic constraint (2.6) also ensures mutual locality of electric and magnetic charges.  $D_\mu M_{\alpha\beta}$  depends only on  $\xi_{\alpha M}$  which, as we will see shortly, vanish for  $N = 4$  AdS backgrounds implying that  $\tau$  is uncharged and  $D_\mu M_{\alpha\beta}$  reduces to an ordinary derivative.

The conditions for a supersymmetric AdS-background can be concisely formulated in terms of the scalar components of the  $N = 4$  supersymmetry transformations. For the four gravitinos  $\psi_\mu^i$ , the four spin-1/2 fermions in the gravitational multiplet  $\chi^i$  and the gauginos  $\lambda_a^i$  they are given by [14]

$$\begin{aligned} \delta\psi_\mu^i &= 2D_\mu\epsilon^i - \frac{2}{3}A_1^{ij}\Gamma_\mu\epsilon_j + \dots \ , \\ \delta\chi^i &= -\frac{4}{3}iA_2^{ji}\epsilon_j + \dots \ , \\ \delta\lambda_a^i &= 2iA_{2aj}{}^i\epsilon^j + \dots \ , \end{aligned} \quad (2.9)$$

where  $\epsilon_j$  are the four supersymmetry parameters and the dots indicate terms that vanish in a maximally symmetric space-time background. The fermion shift matrices read

$$\begin{aligned} A_1^{ij} &= \epsilon^{\alpha\beta}(\nu_\alpha)^*\nu_{kl}^M\nu_N^{ik}\nu_P^{jl}f_{\beta M}{}^{NP} \ , \\ A_2^{ij} &= \epsilon^{\alpha\beta}\nu_\alpha\nu_{kl}^M\nu_N^{ik}\nu_P^{jl}f_{\beta M}{}^{NP} + \frac{3}{2}\epsilon^{\alpha\beta}\nu_\alpha\nu_M^{ij}\xi_\beta^M \ , \\ A_{2ai}{}^j &= \epsilon^{\alpha\beta}\nu_\alpha\nu_a^M\nu_{ik}^N\nu_P^{jk}f_{\beta MN}{}^P - \frac{1}{4}\delta_i^j\epsilon^{\alpha\beta}\nu_\alpha\nu_a^M\xi_{\beta M} \ , \end{aligned} \quad (2.10)$$

where the  $\nu_M^{ij}$  are defined with the help of  $\text{SO}(6)$   $\Gamma$ -matrices as

$$\nu_M^{ij} = \nu_M^m(\Gamma_m)^{ij} \ . \quad (2.11)$$

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<sup>4</sup>In the embedding tensor formalism electric and magnetic gauge bosons are simultaneously introduced into the action and a global  $G = \text{SL}(2) \times \text{SO}(6, n)$  is manifest as long as  $\xi_\alpha^M$  and  $f_{\alpha MNP}$  transform as tensors under  $G$ . Any specific and consistent choice of  $\xi_\alpha^M$  and  $f_{\alpha MNP}$  breaks that symmetry and determines the local gauge group  $G_0 \subset G$ .

We give more details on the  $\Gamma$ -matrices in Appendix A. In terms of the shift matrices the scalar potential is given by

$$V = \frac{1}{2} A_{2ai}{}^j A_{2aj}{}^i + \frac{1}{9} A_2{}^{ij} A_{2ij}^* - \frac{1}{3} A_1{}^{ij} A_{1ij}^* . \quad (2.12)$$

### 3 Structure of $N = 4$ AdS<sub>4</sub> backgrounds

In this section we study  $N = 4$  gauged supergravities that admit a fully supersymmetric AdS<sub>4</sub> background, that is, all sixteen supercharges are left unbroken. The latter requirement demands that the supersymmetry variations (2.9) of  $\chi^i$  and  $\lambda_a^i$  have to vanish in the AdS<sub>4</sub> background while the supersymmetry variations of the gravitinos have to be proportional to the cosmological constant. Inspecting (2.9) and (2.12) we see that this implies

$$\langle A_2^{ij} \rangle = \langle A_{2a}^{ij} \rangle = 0 , \quad \text{and} \quad \langle A_1^{ij} A_{1kj}^* \rangle = |\mu|^2 \delta_k^i , \quad (3.1)$$

where  $\langle V \rangle = -\frac{4}{3}|\mu|^2$  is the cosmological constant and  $\langle \cdot \rangle$  indicates that a quantity is evaluated in the AdS-background. In  $A_2$  the first (second) term is anti-symmetric (symmetric) in  $i$  and  $j$  and thus they have to vanish independently. Similarly in  $A_{2a}$  the two terms correspond to a decomposition into the trace and a traceless part and thus they also have to vanish independently. We can immediately conclude that fully supersymmetric AdS<sub>4</sub> backgrounds can only occur in  $N = 4$  supergravities which have

$$\xi_\alpha^M = 0 . \quad (3.2)$$

This property considerably simplifies the following analysis and is also the reason why  $M_{\alpha\beta}$  or similarly  $\tau$  is uncharged in the Lagrangian (2.7). From (2.2) we also see that for purely electric gaugings, i.e.  $\xi_\alpha^M = f_{-NPQ} = 0$ , one has  $A_1 = A_2$  and thus no supersymmetric AdS-background is possible [10, 11].

Inserting (3.2) into (2.10) the conditions (3.1) simplify and read

$$\langle A_1^{ij} \rangle = \langle \nu_{kl}^M \nu_N^{ik} \nu_P^{jl} \nu_\alpha^* \rangle \epsilon^{\alpha\beta} f_{\beta M}{}^{NP} = \mu P^{ij} , \quad (3.3)$$

$$\langle A_2^{ij} \rangle = \langle \nu_{kl}^M \nu_N^{ik} \nu_P^{jl} \nu_\alpha \rangle \epsilon^{\alpha\beta} f_{\beta M}{}^{NP} = 0 , \quad (3.4)$$

$$\langle A_{2a}^{ij} \rangle = \langle \nu_a^M \nu_{ik}^N \nu_P^{jk} \nu_\alpha \rangle \epsilon^{\alpha\beta} f_{\beta MN}{}^P = 0 , \quad (3.5)$$

where  $P^{ij}$  is a constant matrix obeying  $P^{ik} P_{kj} = \delta_j^i$  but is otherwise arbitrary. The conditions (3.3)–(3.5) have to be solved subject to the quadratic constraints (2.6) which for  $\xi_\alpha^M = 0$  also simplify and are given by

$$f_{\alpha R[MN} f_{\beta|PQ]}{}^R = 0 , \quad \epsilon^{\alpha\beta} f_{\alpha MNR} f_{\beta PQ}{}^R = 0 . \quad (3.6)$$

Due to the homogeneity of the  $N = 4$  field space (2.1) one can translate any point of  $\mathcal{M}$  to its origin and perform the analysis there.<sup>5</sup> Here we prefer to perform the analysis at some arbitrary but fixed vacuum expectation value of the scalar fields corresponding to an  $\text{AdS}_4$  background. This leads us to redefine the components of the embedding tensor and introduce the complex quantities

$$\mathbf{f}_{QRS} = \mathbf{f}_{1QRS} + i \mathbf{f}_{2QRS} = \langle \nu_Q^M \nu_R^N \nu_S^P \nu_\alpha \rangle \epsilon^{\alpha\beta} f_{\beta NMP} , \quad (3.7)$$

where  $\mathbf{f}_{1,2QRS}$  are the real and imaginary parts, respectively. Using (2.2) they are given by

$$\mathbf{f}_{1QRS} = \langle \frac{1}{\sqrt{\text{Im} \tau}} \nu_Q^M \nu_R^N \nu_S^P \rangle (\langle \text{Re} \tau \rangle f_{-NMP} - f_{+NMP}) , \quad \mathbf{f}_{2QRS} = \langle \sqrt{\text{Im} \tau} \nu_Q^M \nu_R^N \nu_S^P \rangle f_{-NMP} . \quad (3.8)$$

We see that  $\mathbf{f}_2$  is directly related to the magnetic components  $f_-$  of the embedding tensor while  $\mathbf{f}_1$  is an admixture of electric and magnetic components. Note that at the origin of  $\mathcal{M}$  the vielbeins are unit matrices,  $\langle \text{Re} \tau \rangle$  vanishes and  $\mathbf{f}_1$  is purely electric. Before we proceed let us also give the quadratic constraint (3.6) in terms of  $\mathbf{f}$ . Using (2.2) one finds

$$\mathbf{f}_{[MN}{}^R \mathbf{f}_{PQ]R} = 0 , \quad \text{Re}(\mathbf{f}_{[MN}{}^R \mathbf{f}_{PQ]R}^*) = 0 , \quad \text{Im}(\mathbf{f}_{MN}{}^R \mathbf{f}_{PQR}^*) = 0 . \quad (3.9)$$

Let us now turn to the solution of the conditions (3.3)–(3.5) and start by analyzing the gaugino variation (3.5). Using (2.11), (3.7) and the antisymmetry of the  $f_{\alpha MNP}$  we can rewrite (3.5) as

$$\mathbf{f}_{amn} (\Gamma^{mn})^{ij} = 0 , \quad (3.10)$$

where  $\Gamma^{mn}$  are generators of  $\text{SU}(4)$  defined in Appendix A. Since the  $\Gamma^{mn}$  are linearly independent generators we immediately conclude

$$\mathbf{f}_{amn} = 0 , \quad (3.11)$$

which, using (3.8), also implies  $f_{\alpha amn} = 0$ .

We employ the same strategy to analyze the variations of the fermions in the gravitational multiplet, i.e. (3.3) and (3.4). Using (2.11) and (3.7) they are equivalent to

$$\mathbf{f}_{nmp}^* (\Gamma^n \Gamma^{*m} \Gamma^p)^{ij} = -\mu P^{ij} , \quad \mathbf{f}_{nmp} (\Gamma^n \Gamma^{*m} \Gamma^p)^{ij} = 0 . \quad (3.12)$$

Since the antisymmetric products of three  $\Gamma$ -matrices are linearly independent up to the relation (A.3), we can further rewrite (3.12) as

$$\mathbf{f}_{mnp} + i \epsilon_{mnpqrs} \mathbf{f}_{qrs} = 0 , \quad \mathbf{f}_{mnp} \mathbf{f}^{*mnp} = 2|\mu|^2 . \quad (3.13)$$

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<sup>5</sup>This has been frequently used, for example in [12, 15, 16].

We learn that  $\mathfrak{f}_{mnp}$  ( $\mathfrak{f}_{mnp}^*$ ) is imaginary self-dual (anti-self-dual) with a norm related to the cosmological constant.<sup>6</sup> In addition to (3.13) the  $\mathfrak{f}_{mnp}$  have to satisfy the quadratic constraints. Due to (3.11) the  $mnpq$ -component of (3.9) simplifies and reads

$$\mathfrak{f}_{[mn}{}^r \mathfrak{f}_{pq]r} = 0, \quad \text{Re}(\mathfrak{f}_{[mn}{}^r \mathfrak{f}_{pq]r}^*) = 0, \quad \text{Im}(\mathfrak{f}_{mn}{}^r \mathfrak{f}_{pqr}^*) = 0. \quad (3.14)$$

In Appendix B we show that (3.13) and (3.14) together have a unique solution which can always be put into the form

$$\mathfrak{f}_{123} = \frac{1}{\sqrt{6}} \mu, \quad \mathfrak{f}_{456} = -\frac{i}{\sqrt{6}} \mu, \quad (3.15)$$

or, in terms of real and imaginary part and for  $\mu$  real<sup>7</sup>

$$\mathfrak{f}_{1\,123} = \frac{1}{\sqrt{6}} \mu, \quad \mathfrak{f}_{2\,456} = -\frac{1}{\sqrt{6}} \mu. \quad (3.16)$$

In terms of the  $N = 4$  gauge theory this implies that an  $\text{SO}(3)_+ \times \text{SO}(3)_-$  subgroup of the R-symmetry group  $\text{SO}(6)_R$  which rotates the six graviphotons into each other has to be gauged in order for a fully supersymmetric AdS-background to exist. One of the factors is electric while the second factor is magnetic.

This concludes our solution of (3.3)–(3.5) as they do not involve the components  $\mathfrak{f}_{mab}$  and  $\mathfrak{f}_{abc}$  of the (redefined) embedding tensor. Or in other words we can choose  $\mathfrak{f}_{mab} = \mathfrak{f}_{abc} = 0$  without affecting the AdS solution. In this case the entire gauge group is

$$G_0 = \text{SO}(3)_+ \times \text{SO}(3)_- \subset \text{SO}(6)_R, \quad (3.17)$$

and the scalar fields are only charged with respect to the six graviphotons while they remain neutral with respect to all other  $n$  Abelian vector fields. The number of vector multiplets  $n$  in this solution is arbitrary including  $n = 0$  in which case  $\tau$  is the only scalar field.<sup>8</sup>

However, as we will show now, fully supersymmetric AdS-backgrounds with larger gauge groups  $G_0$  can also exist for  $\mathfrak{f}_{mab} \neq 0$  and/or  $\mathfrak{f}_{abc} \neq 0$ .<sup>9</sup> In this case the solution (3.15) of (3.3)–(3.5) is unaffected but the quadratic constraints (3.9) change and have to be reanalyzed. In particular they couple different components of the embedding tensor.

Let us first consider supergravities with  $\mathfrak{f}_{mab} = 0, \mathfrak{f}_{abc} \neq 0$ . In this case the quadratic constraints (3.9) split into two disjoint set of conditions and give a standard Jacobi-identity for  $\mathfrak{f}_{abc}$ . Thus the gauge group is

$$G_0 = \text{SO}(3)_+ \times \text{SO}(3)_- \times G_0^v \subset \text{SO}(6, n), \quad (3.18)$$

<sup>6</sup>Note that at the origin the  $\mathfrak{f}_{mnp}$  are related to the  $f^{(\pm)}$  defined in [16]. Furthermore, (3.13) forbids any real  $\mathfrak{f}_{mnp}$  which corresponds to the observation that a purely electric gauge theory does not admit an AdS-background.

<sup>7</sup>By an appropriate rotation of the gravitinos  $\mu$  can always be chosen real.

<sup>8</sup>For  $n = 0, 1$  this solution was first found in [9] and it is also discussed in [15].

<sup>9</sup>Physically the  $f_{\alpha mab}$  determine the supersymmetric fermionic and bosonic mass matrices while  $f_{\alpha abc}$  only contributes to mass terms when supersymmetry is broken [16].



where  $G_0^v \subset \text{SO}(n)$  is the gauge group with structure constants  $f_{\alpha abc}$  which only acts among the gauge bosons of the vector multiplets.

For  $\mathfrak{f}_{mab} \neq 0, \mathfrak{f}_{abc} \neq 0$  the situation is slightly more involved. First of all there can be a split within the  $\mathfrak{f}_{abc}$  into two disjoint sets so that one subset of them has no common indices with  $\mathfrak{f}_{mab}$  and thus satisfies a standard Jacobi-identity with no further interference terms in (3.9). As before this corresponds to a separate factor  $G_0^v \subset \text{SO}(q), q \leq n$  in the gauge group  $G_0$ .

Now let us turn to the  $\mathfrak{f}_{mab}, \mathfrak{f}_{abc}$  which do share common indices  $a, b, c$ . Considering the  $m nab$ -component of the last constraint in (3.9) we learn that in the basis where (3.16) holds,  $\mathfrak{f}_{1mab}$  can only be non-zero for  $m = 1, 2, 3$  while  $\mathfrak{f}_{2mab}$  can only be non-zero for  $m = 4, 5, 6$ . Furthermore the same equation also says that  $\mathfrak{f}_{1mab}$  and  $\mathfrak{f}_{2mab}$  cannot share any  $ab$  indices. Or in other words the  $\mathfrak{f}_{mab}$  decompose into two disjoint sets for  $\mathfrak{f}_{1mab}, m = 1, 2, 3$  and  $\mathfrak{f}_{2nab}, n = 4, 5, 6$ . With this observation all other quadratic constraints turn into standard Jacobi-identities of three separate group factors  $G_+, G_-, G_0^v$  so that the total gauge group is of the form

$$G_0 = G_+ \times G_- \times G_0^v \subset \text{SO}(6, n) , \quad (3.19)$$

where

$$G_+ \subset \text{SO}(3, m_+) , \quad G_- \subset \text{SO}(3, m_-) . \quad (3.20)$$

The maximal compact subgroups for each factor are  $\text{SO}(3)_\pm \times H_\pm$  with  $H_\pm \subset \text{SO}(m_\pm)$  and  $m_+ + m_- + q = n$ . Special cases of this solution have been discussed in [15]. As we will see in the next section in the  $N = 4$  AdS background  $G_0$  is spontaneously broken to its maximal compact subgroup.

## 4 $N = 4$ AdS moduli space

After having determined the  $N = 4$  AdS-backgrounds we turn to the question to what extent they are isolated points in field space or if they can have flat directions (a moduli space) which preserve all supercharges. We use the same method as in [3] in that we vary the supersymmetry conditions (3.1) and then find all possible directions in the scalar field space  $\mathcal{M}$  which are left undetermined by (3.1). More concretely, we look for continuous solutions of

$$\delta A_1^{ij} = \delta A_2^{ij} = \delta A_{2a}^{ij} = 0 , \quad (4.1)$$

in the vicinity of a fully supersymmetric  $\text{AdS}_4$  background.

In order to do so we have to parametrize the variations of the vielbeins. Let us define the  $6n$  scalar field fluctuations  $\delta\phi_{ma}$  around the  $\text{AdS}_4$  background value by

$$\delta\nu_M^m = \langle \nu_M^a \rangle \delta\phi_{ma} . \quad (4.2)$$

Then we find from (2.4) (suppressing henceforth the bracket  $\langle \cdot \rangle$ )

$$\delta \nu_M^a = \nu_M^m \delta \phi_{ma} . \quad (4.3)$$

Similarly, we have for the inverse vielbeins

$$\delta \nu_a^M = -\nu_m^M \delta \phi_{ma} , \quad \delta \nu_m^M = -\nu_a^M \delta \phi_{ma} . \quad (4.4)$$

Thus at linear order in  $\delta \phi$  the metric  $M_{MN}$  is given by

$$M_{MN} = \begin{pmatrix} \delta_{mn} & 2\delta \phi_{mb} \\ 2\delta \phi_{an} & \delta_{ab} \end{pmatrix} + \mathcal{O}(\delta \phi^2) . \quad (4.5)$$

Similarly, for the  $SL(2)/SO(2)$  factor of  $\mathcal{M}$  we have

$$\delta \nu_\alpha = \frac{i}{2\text{Im} \tau} (\nu_\alpha^* \delta \tau - \nu_\alpha \delta \text{Re} \tau) . \quad (4.6)$$

Using (4.4) and (4.6) we can also determine the variations of  $\mathfrak{f}$  to be<sup>10</sup>

$$\begin{aligned} \delta \mathfrak{f}_{npq} &= -3\delta_{m[n} \mathfrak{f}_{pq]a} \delta \phi_{ma} + \frac{1}{\text{Im} \tau} \text{Im} \mathfrak{f}_{npq} \delta \text{Re} \tau - \frac{1}{2\text{Im} \tau} \mathfrak{f}_{npq}^* \delta \text{Im} \tau , \\ \delta \mathfrak{f}_{npb} &= (2\delta_{m[n} \mathfrak{f}_{p]ab} - \delta_{ab} \mathfrak{f}_{mnp}) \delta \phi_{ma} + \frac{1}{\text{Im} \tau} \text{Im} \mathfrak{f}_{npb} \delta \text{Re} \tau - \frac{1}{2\text{Im} \tau} \mathfrak{f}_{npb}^* \delta \text{Im} \tau . \end{aligned} \quad (4.7)$$

With the help of these variations we can now discuss the variations of  $A_1$  and  $A_2$ . Starting from (3.12) and using that  $P^{ij}$  is constant we obtain

$$\delta \mathfrak{f}_{mnp}^* = \delta \mathfrak{f}_{mnp} = 0 . \quad (4.8)$$

Using (4.7) and (3.11) this implies

$$\mu P^{ij} \delta \tau = 0 , \quad (4.9)$$

leaving  $\delta \tau = 0$  as the only solution. Thus the complex scalar  $\tau$  is necessarily fixed in any  $N = 4$  AdS-background.

We are left with the variation of  $A_{2a}$  or in other words the variation of (3.11). Using (4.7) and (4.9) we find

$$\mathfrak{f}_{mnp} \delta \phi_{pa} - 2\mathfrak{f}_{ab[m} \delta \phi_{n]b} = 0 . \quad (4.10)$$

In Appendix C we show that all solutions of (4.10) have to be of the form

$$\delta \phi_{ma} = \mathfrak{f}_{1abm} \lambda_1^b + \mathfrak{f}_{2abm} \lambda_2^b = f_{\alpha abm} \lambda_\alpha^b , \quad (4.11)$$

where  $\lambda_{1,2}^b$  or equivalently  $\lambda_\alpha^b$  are arbitrary real parameters.

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<sup>10</sup>These equations are equivalent to the gradient flow equations given in [6, 17].

The deformations (4.11) also have a geometrical meaning. The Lagrangian and the background are invariant under global symmetry transformations inside  $SO(6, n)$  that leave the gauge group  $G_0 = G_+ \times G_- \times G_0^v$  (and the associated structure constants) invariant. This global symmetry is  $G_0$  itself times its maximal commutant  $H_c$  inside  $SO(6, n)$ .<sup>11</sup> Since  $H_c$  commutes with  $G_\pm$ , it commutes with  $SO(3)_\pm$  and therefore must be inside  $SO(n)$ , i.e.  $H_c$  is a compact group. Thus, the scalar deformations in (2.1) which preserve supersymmetry should correspond to the non-compact directions in  $G_0 \times H_c$ . Since  $G_0^v$  and  $H_c$  are compact, the supersymmetric scalar deformations span the coset

$$\mathcal{M}_{N=4} = \frac{G_+}{H_+} \times \frac{G_-}{H_-} , \quad (4.12)$$

where  $H_\pm$  are the compact subgroups of  $G_\pm$  and contain an  $SO(3)_\pm$  factor. If we linearize the scalars in this coset, we indeed find the deformations (4.11).

Let us confirm the masslessness of the deformations (4.11) by computing their scalar mass matrix. The mass matrices of  $N = 4$  gauged supergravity have been given and analyzed for example in [15, 16]. Nevertheless let us spend a few steps to derive them in an  $N = 4$  background in our notation. Inserting (2.10) into (2.12) using (3.7) one finds for  $\xi_M = 0$

$$\frac{1}{4}V = -\frac{1}{3}\mathbf{f}_{mnp}(\delta^{mq}\delta^{nr}\delta^{ps} + i\epsilon^{mnpqrs})\mathbf{f}_{qrs}^* + \frac{1}{2}\mathbf{f}_{amn}\mathbf{f}_{amn}^* + \frac{1}{9}\mathbf{f}_{mnp}(\delta^{mq}\delta^{nr}\delta^{ps} - i\epsilon^{mnpqrs})\mathbf{f}_{qrs}^* . \quad (4.13)$$

Computing the second derivative of  $V$  with respect to  $\delta\phi_{am}$ , we find for an  $N = 4$  background (where (3.11) and (3.13) hold) the mass matrix

$$\begin{aligned} M_{am,bn} &= -16\delta_{ab}\text{Re}(\mathbf{f}_{mpq}\mathbf{f}_{npq}^*) + 4\text{Re}[(2\delta_{m[p}\mathbf{f}_{q]ac} + \delta_{ac}\mathbf{f}_{mpq})(2\delta_{n[p}\mathbf{f}_{q]bc} + \delta_{bc}\mathbf{f}_{npq}^*)] \\ &= -16\delta_{ab}f_{\alpha mpq}f_{\alpha npq} + 4(\delta_{ac}f_{\alpha mpq} + 2f_{\alpha ca[p}\delta_{q]m})(\delta_{bc}f_{\alpha npq} + 2f_{\alpha cb[p}\delta_{q]n}) . \end{aligned} \quad (4.14)$$

We see that scalars of the form (4.11) indeed fulfill  $M_{am,bn}\delta\phi_{bn} = 0$  and thus are massless. This confirmation can also be viewed as a consistency check of (4.10).<sup>12</sup> It is easy to see from (4.13) that there are no mass terms mixing  $\delta\tau$  and  $\delta\phi_{am}$  for an  $N = 4$  vacuum.

Let us finally show that all flat directions in (4.11) correspond to Goldstone bosons which are eaten by massive vector fields. To do so we inspect the covariant derivative of  $\delta\phi_{ma}$  and find from (2.8) and (4.5)

$$D_\mu\delta\phi_{am} = \partial_\mu\delta\phi_{am} + \mu(A_\mu^{+p} - A_\mu^{-p})\epsilon_{pmn}\delta\phi_{an} + (A_\mu^{\alpha n}f_{\alpha nab} + A_\mu^{\alpha c}f_{\alpha cab})\delta\phi_{bm} + 2A_\mu^{\alpha b}f_{\alpha bam} + \mathcal{O}(\delta\phi^2) . \quad (4.15)$$

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<sup>11</sup>Among the  $Sl(2, \mathbb{R})$  transformations only  $SO(2)$  preserves the electric and magnetic gaugings, which is a compact generator and therefore does not give any massless deformations.

<sup>12</sup> Note that since (4.14) is a sum of squares that come with different signs, there can be more massless scalars. Examples of this can be found for instance in [15]. However, such scalars do not preserve supersymmetry and will have a potential beyond quadratic order.

First of all we note that in the AdS-background, i.e. for  $\delta\phi_{am} = 0$ , all six graviphotons  $A^{\alpha m}$  are massless and thus, as expected, the  $\text{SO}(3)_+ \times \text{SO}(3)_-$  part of the gauge symmetry is unbroken. From the last term we see that there is a mass term for the gauge bosons  $A_\mu^{\alpha a}$  in the vector multiplets given by  $\hat{M}_{\alpha a \beta b}^2 \sim f_{\alpha a c m} f_{\beta b c m}$ . This gives mass to  $\text{rk}(\hat{M})$  gauge bosons where  $\hat{M}_{ab}^{\alpha m} \sim f_{\alpha m a b}$ . This coincides with the number of flat directions determined in (4.11) as the same matrix  $\hat{M}$  appears. This is no coincidence: When the gauge group is spontaneously broken from  $G_0 = G_+ \times G_- \times G_0^v \rightarrow H_+ \times H_- \times G_0^v$ , the Goldstone bosons form the coset (4.12). Therefore the supersymmetric directions in (4.12) are precisely the Goldstone bosons eaten by the massive vectors. Thus we showed that all supersymmetric scalar deformations are Goldstone bosons and therefore any  $\text{AdS}_4$  background that preserves all supercharges is an isolated point in field space with no further moduli.

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# Appendix

## A SU(4) $\Gamma$ -matrix properties

The fermions of  $N = 4$  gauged supergravity transform in the fundamental representation of the SU(4) R-symmetry. On the scalars the R-symmetry acts as  $\text{SO}(6) \sim \text{SU}(4)/\mathbb{Z}_2$  rotations. The two representations are linked via the  $\Gamma$ -matrices  $\Gamma_m^{ij} = \Gamma_m^{[ij]}$  given by [16]

$$\begin{aligned} \Gamma_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \\ \Gamma_4 &= \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (\text{A.1})$$

They obey

$$\{\Gamma_m, \Gamma_n^*\} = 2\delta_{mn}\mathbf{1}, \quad (\Gamma_m)_{ij} = (\Gamma_m^{ij})^* = \frac{1}{2}\epsilon_{ijkl}\Gamma_m^{kl}. \quad (\text{A.2})$$

The antisymmetric products of two  $\Gamma$ -matrices  $(\Gamma_{mn})_j^i = \frac{1}{2}(\Gamma_m)^{ik}(\Gamma_n^*)_{kj}$  are the (linearly independent) generators of SU(4). On the other hand the antisymmetric products of three  $\Gamma$ -matrices obey the relation

$$(\Gamma_{[m})^{ik}(\Gamma_n)_{kl}(\Gamma_{p]})^{jl} = i\epsilon_{mnpqrs}(\Gamma_q)^{ik}(\Gamma_r)_{kl}(\Gamma_s)^{jl}. \quad (\text{A.3})$$

In (2.11) we use the  $\Gamma$ -matrices to convert the  $\text{SO}(6, n)$  vielbein components  $\nu_M^m$  into objects with spinor indices, i.e. we define  $\nu_M^{ij} = \nu_M^m \Gamma_m^{ij}$ .

## B Classification of structure constants

In this appendix we supply the details of the solution of (3.13) and (3.14) given in (3.15). We know from (3.13) that  $\mathfrak{f}$  is the coefficient of an imaginary self-dual three-form, which means that the form is of type  $(2, 1) \oplus (0, 3)$  with respect to a given complex structure  $I$  on the six-dimensional space parametrized by the index  $m$ . In addition we have to solve the quadratic constraints (3.14) for such a three-form. If we write  $\mathfrak{f}$  with holomorphic and

anti-holomorphic indices  $u, \bar{u} = 1, 2, 3$ , the quadratic constraints can be rewritten as

$$\begin{aligned} \mathfrak{f}_{[\bar{u}\bar{v}|\bar{x}}\delta^{\bar{x}v}\mathfrak{f}_{uv|\bar{w}}] &= 0, & \mathfrak{f}_{[uv|\bar{v}}\delta^{\bar{v}x}\mathfrak{f}_{w]x\bar{u}} &= 0, & \mathfrak{f}_{\bar{u}\bar{v}\bar{y}}\delta^{\bar{y}u}\mathfrak{f}_{u\bar{w}\bar{x}}^* - \mathfrak{f}_{\bar{u}\bar{v}u}^*\delta^{u\bar{y}}\mathfrak{f}_{\bar{w}\bar{x}\bar{y}} &= 0, \\ \mathfrak{f}_{uv\bar{w}}\delta^{\bar{w}w}\mathfrak{f}_{\bar{u}\bar{v}w}^* - \mathfrak{f}_{uvw}^*\delta^{w\bar{w}}\mathfrak{f}_{\bar{u}\bar{v}\bar{w}} &= 0, & \mathfrak{f}_{wu\bar{u}}\delta^{w\bar{w}}\mathfrak{f}_{\bar{w}\bar{v}v}^* - \mathfrak{f}_{u\bar{u}\bar{w}}^*\delta^{\bar{w}w}\mathfrak{f}_{\bar{v}vw} &= 0, \\ 2\mathfrak{f}_{w[u\bar{u}}\delta^{w\bar{w}}\mathfrak{f}_{\bar{v}]v\bar{w}}^* + 2\mathfrak{f}_{\bar{w}[u\bar{u}}^*\delta^{\bar{w}w}\mathfrak{f}_{\bar{v}]v\bar{w}} &+ \mathfrak{f}_{uv\bar{w}}\delta^{\bar{w}w}\mathfrak{f}_{\bar{u}\bar{v}w}^* + \mathfrak{f}_{uvw}^*\delta^{w\bar{w}}\mathfrak{f}_{\bar{u}\bar{v}\bar{w}} &= 0. \end{aligned} \quad (\text{B.1})$$

In terms of a more convenient parametrization

$$\mathfrak{f}_{uv\bar{u}} = \epsilon_{uvw}\delta_{\bar{u}x}\alpha^{wx}, \quad \mathfrak{f}_{\bar{u}\bar{v}\bar{w}} = \epsilon_{\bar{u}\bar{v}\bar{w}}\beta, \quad (\text{B.2})$$

(B.1) reads

$$\begin{aligned} \beta\alpha^{[uv]} &= 0, & \epsilon_{vw\bar{x}}\alpha^{vu}\alpha^{wx} &= 0, & \alpha^{uv}\delta_{v\bar{v}}(\alpha^*)^{\bar{u}\bar{v}} &= |\beta|^2\delta^{u\bar{u}}, \\ \alpha^{vu}\delta_{v\bar{v}}(\alpha^*)^{\bar{v}\bar{u}} + \epsilon^{uvw}(\alpha^*)_{vw}\epsilon^{\bar{u}\bar{v}\bar{w}}\alpha_{\bar{v}\bar{w}} &= |\beta|^2\delta^{u\bar{u}}, \\ \alpha^{uv}(\alpha^*)^{\bar{v}\bar{u}} - \alpha^{vu}(\alpha^*)^{\bar{u}\bar{v}} - \delta^{u\bar{v}}\delta_{w\bar{w}}\alpha^{wv}(\alpha^*)^{\bar{w}\bar{u}} + \delta^{\bar{u}v}\delta_{w\bar{w}}\alpha^{wu}(\alpha^*)^{\bar{w}\bar{v}} &= 0. \end{aligned} \quad (\text{B.3})$$

As we will now show these constraints imply that  $\alpha$  is symmetric. To see this, assume that the antisymmetric part of  $\alpha$  is non-zero and parametrize it by  $\alpha^{[uv]} = \epsilon^{uvw}a_w$ . From the last two equations of (B.3) we then find

$$a_u(\alpha^*)^{(\bar{u}\bar{v})} = 2\epsilon_{uvw}\delta^{v\bar{w}}a_{\bar{w}}^*\delta^{x(\bar{u}}\delta^{\bar{v})w}, \quad (\text{B.4})$$

which after contraction with  $\bar{a}_{\bar{u}}$  and using the second equation of (B.3) shows that  $a_u = 0$  and  $\alpha$  therefore is symmetric. With this simplification the remaining conditions of (B.3) imply

$$\alpha^{uv}(\alpha^*)_{wv} = |\beta|^2\delta_v^u, \quad (\text{B.5})$$

where we lowered the indices with  $\delta_{u\bar{u}}$ . Since (B.5) says that the symmetric matrix  $\alpha$  is also normal, we can diagonalize it as a complex matrix, and the diagonal entries are  $\beta$  times some phase factors. Rotating the holomorphic coordinates by a phase corresponds to  $\text{SO}(2)^3 \subset \text{SO}(6)$  rotations, and together with electromagnetic  $\text{SO}(2) \subset \text{SL}(2)$  rotations that only affect the overall phase of  $\mathfrak{f}$ , we can bring  $\alpha$  and  $\beta$  into the final form

$$\beta = \frac{1}{4\sqrt{6}}\mu, \quad \alpha^{uv} = \frac{1}{4\sqrt{6}}\mu\delta^{uv}, \quad (\text{B.6})$$

where we also used the second equation in (3.13). If we choose the complex structure as

$$I = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\text{B.7})$$

we find for the embedding tensor from (B.2)

$$\mathbf{f}_{123} = \frac{1}{\sqrt{6}} \mu , \quad \mathbf{f}_{456} = -\frac{i}{\sqrt{6}} \mu . \quad (\text{B.8})$$

This indeed says that the gauge group of the graviphotons is  $\text{SO}(3)_+ \times \text{SO}(3)_- \subset \text{SO}(6)$ .

## C $\text{SO}(3)$ group theory

In this appendix we determine the solution of (4.10). First let us record the  $mnab$ -component of the quadratic constraint (3.9)

$$\begin{aligned} 2\mathbf{f}_{[ma}{}^c \mathbf{f}_{bn]c} - \mathbf{f}_{mn}{}^p \mathbf{f}_{abp} &= 0 , & 2\text{Re}(\mathbf{f}_{[ma}{}^c \mathbf{f}_{bn]c}^*) - \text{Re}(\mathbf{f}_{mn}{}^p \mathbf{f}_{abp}^*) &= 0 , \\ \text{Im}(\mathbf{f}_{mn}{}^r \mathbf{f}_{abr}^*) &= 0 , & \text{Im}(\mathbf{f}_{ma}{}^c \mathbf{f}_{nbc}^*) &= 0 . \end{aligned} \quad (\text{C.1})$$

As we already noted in Section 3, the last but one equation together with (3.15) implies that  $\mathbf{f}_{1mab}$  is only non-zero for  $m = 1, 2, 3$  while  $\mathbf{f}_{2mab}$  is only non-zero for  $m = 4, 5, 6$ . This in turn simplifies (C.1) further and implies the two decoupled equations

$$\mathbf{f}_{1,2mac} \mathbf{f}_{1,2ncb} - \mathbf{f}_{1,2nac} \mathbf{f}_{1,2mcb} = \mathbf{f}_{1,2mn}{}^p \mathbf{f}_{1,2pab} , \quad (\text{C.2})$$

for the real and imaginary part of  $\mathbf{f}_{mab}$ . From (C.2) we learn that the  $\mathbf{f}_{1,2mab}$  act on the  $a$ -type indices as  $\text{SO}(3)$  matrices in some (possibly reducible) representations.

With these preliminaries let us return to the solution of (4.10). Since the real and imaginary parts of (4.10) have to vanish independently, we can instead of (4.10) solve

$$\mathbf{f}_{1,2mnp} \delta\phi_{pa} - 2\mathbf{f}_{1,2ab[m} \delta\phi_{n]b} = 0 . \quad (\text{C.3})$$

Note again that due to (3.15) the equation (C.3) for  $\mathbf{f}_1$  gives a constraint for  $\delta\phi_{ma}$  for  $m = 1, 2, 3$  while the equation with  $\mathbf{f}_2$  gives a constraint for  $\delta\phi_{ma}$  with  $m = 4, 5, 6$ . We therefore discuss the solution of (C.3) only for  $\mathbf{f}_1$  and then straightforwardly translate the result for  $\mathbf{f}_2$ . In the following we thus omit the index 1, 2.

Since the linear equation (C.3) is  $\text{SO}(3)$  covariant it projects  $\delta\phi_{ma}$  onto one or several  $\text{SO}(3)$  representations. Let us pick a subspace of the  $n$ -dimensional vector space labeled by  $a$  such that it forms one irreducible  $m_i$ -dimensional representation of  $\text{SO}(3)$  (which we denote by  $\mathbf{m}_i$ ), whose action is given by  $\mathbf{f}_{mab}$  restricted to this subspace. We label this subspace by the index  $\tilde{a} = 1, \dots, m_i$ , so that  $\mathbf{f}_{1m\tilde{a}\tilde{b}}$  acts on it transitively. For this representation, (C.3) reads

$$\mathbf{f}_{m\tilde{a}\tilde{b}} \delta\phi_{n\tilde{b}} - \mathbf{f}_{n\tilde{a}\tilde{b}} \delta\phi_{m\tilde{b}} = \mathbf{f}_{mnp} \delta\phi_{p\tilde{a}} . \quad (\text{C.4})$$

Before we continue, let us recall some properties of  $\text{SO}(3)$  representations. We denote the spin  $s$  representation with dimension  $m = 2s + 1$  by  $\mathbf{m}$ . For the problem at hand we

only need to consider vector-like representations where  $s$  is an integer. The representation  $\mathbf{m}$  can then be understood as a totally symmetric and traceless tensor of degree  $s$ . They can be generated by tensor products of the form

$$\mathbf{3} \otimes \mathbf{m} = \mathbf{m} + \mathbf{2} \oplus \mathbf{m} \oplus \mathbf{m} - \mathbf{2} , \quad \text{for } m \geq 2 , \quad (\text{C.5})$$

which are the symmetric traceless, anti-symmetric and trace components of this tensor product, respectively. A vector  $v^{\tilde{a}}$  in the  $\mathbf{m}$ -representation can be written as  $v_{n_1 \dots n_s} = l_{n_1 \dots n_s}^{\tilde{a}} v^{\tilde{a}}$ ,  $n_1, \dots, n_s = 1, 2, 3$  where  $l_{n_1 \dots n_s}^{\tilde{a}}$  is a constant symmetric, traceless tensor, i.e. it obeys  $l_{n_1 \dots n_s}^{\tilde{a}} = l_{(n_1 \dots n_s)}^{\tilde{a}}$  and  $l_{pq n_1 \dots n_{s-2}}^{\tilde{a}} \delta^{pq} = 0$ . The  $\text{SO}(3)$  action then reads

$$\mathbf{f}_{m\tilde{a}\tilde{b}} l_{n_1 \dots n_s}^{\tilde{b}} = -s \mathbf{f}_{mp(n_1} l_{n_2 \dots n_s)p}^{\tilde{a}} . \quad (\text{C.6})$$

This in particular means that  $\text{SO}(3)$  acts on the  $\mathbf{3}$ -representation via the generators  $-\mathbf{f}_{mnp}$ . From this definition of the  $\mathbf{f}_{m\tilde{a}\tilde{b}}$  and (3.15) we also find

$$\mathbf{f}_{m\tilde{a}\tilde{c}} \mathbf{f}_{m\tilde{c}\tilde{b}}^* = \frac{1}{6} s(s+1) |\mu|^2 \delta_{\tilde{a}\tilde{b}} . \quad (\text{C.7})$$

In this notation the representations (C.5) are then given by

$$\begin{aligned} (w_m v^{\tilde{a}})_{s+1} &= l_{n_1 \dots n_s}^{\tilde{a}} l_{(n_1 \dots n_s}^{\tilde{b}} w_m) v^{\tilde{b}} - \frac{1}{3} s l_{mn_1 \dots n_{s-1}}^{\tilde{a}} l_{n_1 \dots n_{s-1}p}^{\tilde{b}} w_p v^{\tilde{b}} , \\ (w_m v^{\tilde{a}})_s &= \frac{1}{s(s+1)} \mathbf{f}_{mn_1 p} l_{n_2 \dots n_s p}^{\tilde{a}} \mathbf{f}_{qr(n_1} l_{n_2 \dots n_s)r}^{\tilde{b}} w_q v^{\tilde{b}} = \frac{1}{s(s+1)} \mathbf{f}_{m\tilde{a}\tilde{c}} \mathbf{f}_{n\tilde{c}\tilde{b}}^* w_n v^{\tilde{b}} , \\ (w_m v^{\tilde{a}})_{s-1} &= \frac{1}{3} s l_{mn_1 \dots n_{s-1}}^{\tilde{a}} l_{n_1 \dots n_{s-1}p}^{\tilde{b}} w_p v^{\tilde{b}} . \end{aligned} \quad (\text{C.8})$$

After this interlude let us return to the solution of (C.4). If the representation is trivial, i.e. we have  $m_i = 1$  and  $\mathbf{f}_{m\tilde{a}\tilde{b}} = 0$ , we find that (C.4) implies  $\delta\phi_{m\tilde{a}} = 0$ , or in other words we do not find any moduli in this subspace. Let us therefore assume in the following that the representation  $\mathbf{m}_i$  is non-trivial. From their index structure we see that the scalars  $\delta\phi_{m\tilde{a}}$  are in the tensor product (C.5). We thus evaluate the condition (C.3) for each representation in this tensor product individually. If one uses (C.8) and (C.2), one can easily show that the anti-symmetric  $\mathbf{m}_i$ -representation obeys (C.3). If  $\delta\phi_{am}$  is in the  $\mathbf{m}_i + \mathbf{2} \oplus \mathbf{m}_i - \mathbf{2}$  representation we contract (C.4) with  $\mathbf{f}_{mnq}$  and find from (3.15) and (C.2) that

$$\text{Re} (\mathbf{f}_{m\tilde{a}\tilde{c}} \mathbf{f}_{p\tilde{c}\tilde{b}}^* - \mathbf{f}_{p\tilde{a}\tilde{c}} \mathbf{f}_{m\tilde{c}\tilde{b}}^*) \delta\phi_{p\tilde{b}} = \frac{1}{6} |\mu|^2 \delta\phi_{m\tilde{a}} . \quad (\text{C.9})$$

From (C.8) we see that a  $\delta\phi_{m\tilde{a}}$  in the  $\mathbf{m}_i + \mathbf{2} \oplus \mathbf{m}_i - \mathbf{2}$  representation can be written as

$$\delta\phi_{m\tilde{a}} = l_{n_1 \dots n_s}^{\tilde{a}} \delta\tilde{\phi}_{mn_1 \dots n_s} , \quad (\text{C.10})$$

where  $\tilde{\phi}$  is totally symmetric. Using (C.6) one can then show that  $\mathbf{f}_{p\tilde{c}\tilde{b}} \delta\phi_{p\tilde{b}} = 0$  and that  $\mathbf{f}_{p\tilde{a}\tilde{c}} \mathbf{f}_{m\tilde{c}\tilde{b}} \delta\phi_{p\tilde{b}} = -\frac{1}{6} s |\mu|^2 \delta\phi_{m\tilde{a}}$ . Thus, we can conclude that a  $\delta\phi_{ma}$  in the  $\mathbf{m}_i + \mathbf{2}$  and  $\mathbf{m}_i - \mathbf{2}$



representations cannot fulfill (C.4). Or in other words, the condition (C.4) projects onto the  $\mathbf{m}_i$ -representation and we can parametrize  $\delta\phi_{m\bar{a}} = \mathbf{f}_{m\bar{a}\bar{b}}\lambda^{\bar{b}}$  with  $\lambda^{\bar{b}}$  being any element in that representation.

To summarize, we just showed that the scalar deformations  $\delta\phi_{m\bar{a}}$  that preserve  $N = 4$  supersymmetry must be of the form

$$\delta\phi_{ma} = \mathbf{f}_{1abm}\lambda_1^b + \mathbf{f}_{2abm}\lambda_2^b, \quad (\text{C.11})$$

corresponding to  $\text{rk}(\hat{M}) = \sum_i m_{1i} + \sum_i m_{2i}$  massless degrees of freedom, where  $\hat{M}_{ab}^{am} \sim f_{amab}$ .

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